MOTION OF A SOLID BODY WITH A CAVITY CONTAINING AN IDEAL FLUID AND AN AIR BUBBLE

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The motion of a solid body with a cavity partly filled with fluid has usually been studied in those cases where the displacements of the free surface are small relative to the cavity. Rumiantsev [1] for example, gives references to this problem. In this work it is assumed that the fluid almost completely fills the cavity and that it has a nearly spherical air bubble. The problem consists in determination of the motion of the fluid and the bubble, as well as determination of the motion of the solid body containing the cavity.

Let a be the bubble radius, ρ and μ the density and viscosity of the fluid, respectively, σ the coefficient of surface tension at the boundary of fluid and bubble, and v a characteristic value of the fluid velocity relative to the cavity. The effects of viscosity on the motion may be neglected if $\rho va \gg \mu$. Deviation of the bubble shape from spherical will be small if the dynamic addition to the pressure (of the order of ρv^2) is small by comparison with the pressure σ/a specifying the surface tension. Both of the conditions

$$va \gg \mu / \rho, \quad v^2 a \ll \sigma / \rho$$

will be considered as fulfilled. They are satisfied for many fluids, for water in particular, over a wide range of values of v and a. Under the above conditions the bubble can be considered as an undeformable sphere of radius a, and the fluid as an ideal fluid.

Such a postulation does not permit consideration of the motion when the bubble comes into contact with the cavity walls and loses its spherical shape.

The equations of motion are here set up and certain examples are investigated.

1. The motion of a solid body B with a singly connected cavity D, bounded by a wall surface S, is considered. An ideal incompressible fluid of density ρ is inside the cavity together with an undeformable mobile sphere E of radius a and mass m (for a bubble one may put $m = 0$). The ratio $a/l = \varepsilon$, where l is the minimal distance from the center P of the sphere E to the surface S of the walls; this ratio is taken to be small: $\varepsilon \ll 1$.

Let the center of inertia of the sphere be at its geometric center P. Since the fluid is ideal, motion of the sphere relative to its center P

does not affect motion of the fluid and the body B, and from this point of view it is not substantial. Below, only motion of the body B, of the fluid and of the center of the sphere will be studied, and so without loss of generality, motion of the sphere E is considered translational and the forces acting on the sphere are replaced by their principal vector applied at P.

Let the coordinate system $Ox_1x_2x_3$ be rigidly connected to the body, $\mathbf{R}$ and $\mathbf{r}$ are radii-vectors of any point from an immovable and from a movable pole O, respectively.

We consider the body forces acting on the fluid to have a potential $U(\mathbf{r}, t)$, then the potential energy of the fluid is

$$\Pi = \rho\int_D U dv - \rho\int_E U dv \tag{1.1}$$

The fluid flow is assumed to be a potential flow with a velocity potential $\varphi(\mathbf{r}, t)$. The function $\varphi(\mathbf{r}, t)$ is harmonic in the region D_1, occupied by the fluid and bounded by the wall surfaces S and from within by the surface Σ of the sphere E. The boundary conditions

$$\frac{\partial\varphi}{\partial n} = (\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r})\,\mathbf{n} \quad \text{on } S, \qquad \frac{\partial\varphi}{\partial\nu} = \mathbf{v}_p \boldsymbol{\nu} \quad \text{on } \Sigma \tag{1.2}$$

are satisfied by the function φ on both surfaces.

Here $\mathbf{n}$ and $\boldsymbol{\nu}$ are the unit outer normals to the surfaces S and Σ, respectively (Fig.1), $\mathbf{v}_p$ is the absolute velocity of the point P, $\mathbf{v}_0$ the absolute velocity of the pole O, and $\boldsymbol{\omega}$ is the absolute angular velocity of the solid body.

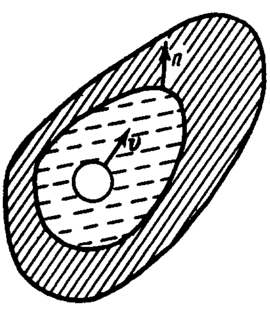

Fig. 1

The position of the system consisting of the body B, the fluid, and the sphere E, will be determined by the radius-vector $\mathbf{R}_0$ of the pole O, the radius-vector $\mathbf{r}_p$ of the point P relative to O ($\mathbf{r}_p = \mathbf{R}_p - \mathbf{R}_0$) and three parameters γ_i ($i = 1,2,3$) giving the angular position of the solid body (for example, the Euler angles). The quantities introduced are connected with the velocities $\mathbf{v}_0$, $\mathbf{v}_p$ and $\boldsymbol{\omega}$ by the kinematic relations (primes denote differentiation with respect to time t in the movable system $Ox_1x_2x_3$)

$$d\mathbf{R}_0 / dt = \mathbf{v}_0, \qquad \mathbf{v}_p = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_p + \mathbf{r}_p' \tag{1.3}$$

and also by three equations connecting the projections of $\boldsymbol{\omega}$ with the parameters γ_i (altogether 9 scalar equations for the 18 scalar variables $\mathbf{R}_0$, $\mathbf{r}_p$, γ_i, $\mathbf{v}_0$, $\mathbf{v}_p$, $\boldsymbol{\omega}$). By means of these variables (coordinates and velocities) and by the present radius-vector $\mathbf{r}$ one may evidently express the position and velocity of any point of the body B and sphere E (the latter is in translational motion). In addition, the velocity potential $\varphi(\mathbf{r}, t)$

subjected to conditions (1.2) and, consequently, the velocity distribution in the fluid, are expressed by these variables. Thus, the quantity of motion $\mathbf{Q}$, the kinetic moment $\mathbf{K}$ relative to the point O, and the kinetic energy T of the system: body B plus fluid plus sphere E, at any moment of time may be expressed as functions of $\mathbf{R}_0$, $\mathbf{r}_p$, γ_i, $\mathbf{v}_0$, $\mathbf{v}_p$ and $\boldsymbol{\omega}$. We note the relations

$$\mathbf{Q} = \partial T / \partial \mathbf{v}_0, \qquad \mathbf{K} = \partial T / \partial \boldsymbol{\omega} \tag{1.4}$$

which may be easily obtained, for example, from those given in Chapter 9 of [2]. Formulas (1.4) hold if T is considered to be a function of $\mathbf{R}_0$, $\mathbf{r}_p$, γ_i, $\mathbf{v}_0$, $\boldsymbol{\omega}$, and $\mathbf{r}_p'$ ($\mathbf{v}_p$ is excluded by means of (1.3)).

We write the equations of motion of the system in the form

$$d\mathbf{Q} / dt = \mathbf{F}, \qquad d\mathbf{K} / dt + \mathbf{v}_0 \times \mathbf{Q} = \mathbf{m}_0 \tag{1.5}$$

where $\mathbf{F}$ is the principal vector of all external forces acting on the system, $\mathbf{m}_0$ the principal moment of those forces relative to the point O. We add to (1.5) the equation of motion of the sphere E, which we write in Lagrangian form

$$\frac{d}{dt}\frac{\partial T}{\partial \mathbf{v}_p} - \frac{\partial T}{\partial \mathbf{R}_p} = \mathbf{Q}_p \tag{1.6}$$

In Equation (1.6) the energy T is considered to be a function of $\mathbf{R}_0$, $\mathbf{R}_p$, γ_i, $\mathbf{v}_0$, $\boldsymbol{\omega}$ and $\mathbf{v}_p$ in distinction from (1.4). The generalized force $\mathbf{Q}_p$ may be presented in the form

$$\mathbf{Q}_p = \mathbf{F}_p - \partial \Pi / \partial \mathbf{R}_p \tag{1.7}$$

Here $\mathbf{F}_p$ is the principal vector of the external forces acting on the sphere (not connected with fluid pressure), Π is the potential energy of the fluid (1.1); the potential energy of the body B is considered independent of the position of the sphere E.

Thus, if T, $\mathbf{Q}$ and $\mathbf{K}$ are found as functions of the variables $\mathbf{R}_0$, $\mathbf{r}_p$ (or $\mathbf{R}_p$), γ_i, $\mathbf{v}_0$, $\boldsymbol{\omega}$, $\mathbf{v}_p$ (or $\mathbf{r}_p'$), then Equations (1.5) and (1.6) may be formed, and together with the kinematic relations they represent a closed system. It is supposed that $\mathbf{F}$, $\mathbf{m}_0$ and $\mathbf{F}_p$ are expressed in the same variables.

The momentum of the whole system is determined by the formula

$$\mathbf{Q} = \mathbf{Q}^\circ + (m - \varrho\Omega)\,\mathbf{v}_p \tag{1.8}$$

where $\mathbf{Q}^\circ$ is the momentum of a system consisting of the solid body with the cavity D completely filled with fluid. Quantities with the superscript $^\circ$ relate to this system and are considered to be known functions of the coordinates and velocities of the solid body.

The kinetic energy of the system

$$T = T_1 + T_2 + {}^1/_2 m v_p^2 \tag{1.9}$$

where T_1 is the known expression for the kinetic energy of a solid body [2] and T_2 is the kinetic energy of the fluid

$$T_2 = \frac{1}{2}\rho\int_{D_1}(\nabla\varphi)^2\,dv = \frac{1}{2}\rho\left(\oint_S \varphi\frac{\partial\varphi}{\partial n}\,ds - \oint_\Sigma \varphi\frac{\partial\varphi}{\partial\nu}\,ds\right) \qquad (1.10)$$

while the last term in (1.9) is the kinetic energy of the sphere E.

Thus, it is necessary to find the potential φ, to calculate T_2 from (1.10), T and $\mathbf{K}$ from (1.9) and (1.4), and $\mathbf{Q}$ from (1.8).

2. The Neumann problem for the function φ, harmonic in the region D_1 and satisfying the boundary conditions (1.2), will be solved by the alternative Schwarz method. We seek the potential φ in the form of an infinite series

$$\varphi = \varphi^\circ + \varphi^1 + \varphi^2 + \ldots \qquad (2.1)$$

where the φ^{2k} are functions harmonic in D (everywhere inside S) and the φ^{2k+1} are functions harmonic outside the sphere E ($k = 0, 1, 2, \ldots$). The function φ° satisfies the first condition (1.2) and is the flow potential of the fluid when the cavity is completely filled. The function φ^1 satisfies the auxiliary condition

$$\partial\varphi^1 / \partial\nu = \mathbf{v}_p\boldsymbol{\nu} - \partial\varphi^\circ / \partial\nu \quad \text{on } \Sigma \qquad (2.2)$$

so that the sum of φ° and φ^1 satisfies exactly the second condition of (1.2). Further, terms of the series satisfy the boundary conditions

$$\partial\varphi^{2k} / \partial n = -\,\partial\varphi^{2k-1} / \partial n \quad \text{on } S \qquad (k = 1, 2, \ldots) \qquad (2.3)$$

for the functions φ^{2k} and the conditions

$$\partial\varphi^{2k+1} / \partial\nu = -\,\partial\varphi^{2k} / \partial\nu \quad \text{on } \Sigma \qquad (k = 1, 2, \ldots) \qquad (2.4)$$

for the functions φ^{2k+1}. The Neumann problem for the functions φ^{2k+1}, harmonic outside the sphere E, is solved in an elementary manner, while the problem for the φ^{2k} may be solved effectively if the Green function for the Neumann problem is known in the region D. The convergence of the alternate Schwarz method (series (2.1)) for the Dirichlet problem has been proved for regions of a very general form [3]; for the Neumann problem in the given case, convergence apparently holds as well. If the series converges, then it is obvious that conditions (1.2) are fulfilled. We note that the Schwarz method may also be applied in the case when E is not a sphere but any other body for which the external Neumann problem can be solved.

We write a solution for the functions φ^{2k+1} satisfying condition

$$\partial\varphi^{2k+1} / \partial\nu = -\,\partial u / \partial\nu \qquad (2.5)$$

on Σ, where $u = \varphi^{2k}$ for $k \geqslant 1$ (2.4) and $u = \varphi^\circ - \mathbf{v}_p\mathbf{r}$ for $k = 0$ (2.2). We introduce a coordinate system $Py_1y_2y_3$ with origin at the point P and axes parallel to those in the $Ox_1x_2x_3$ system. Let ν_i be the projection of the unit normal $\boldsymbol{\nu}$ on the y_i-axis; then evidently $y_i = \nu_i a$ on Σ. We expand the right-hand side of (2.5) in a Taylor series with center at the point P

$$\frac{\partial\varphi^{2k+1}}{\partial\nu} = -\sum_i \nu_i\frac{\partial u}{\partial x_i} = -\frac{1}{a}\left(\sum_i u_i y_i + \frac{1}{1!}\sum_{ij} u_{ij}y_iy_j + \frac{1}{2!}\sum_{ijk} u_{ijk}y_iy_jy_k + \ldots\right) \qquad (2.6)$$

Here u_i, u_{ij}, etc., are partial derivatives of u with respect to x_i, x_i and x_j, etc., taken at the point P and summed over the indices $i, j, \ldots$ from 1 to 3. The sums in (2.6) represent successive terms in the expansion of the harmonic function in Taylor series and so are homogeneous harmonic polynomials [4]. On the sphere Σ they may be written in the form (the Y_n is a spherical function)

$$\frac{1}{(n-1)!}\sum_{(n)} u_{ij\ldots k}y_i y_j \ldots y_k = \frac{a^n}{(n-1)!}\sum_{(n)} u_{ij\ldots k}\nu_i\nu_j \ldots \nu_k = Y_n$$

Here the symbol (n) denotes the number of indices $i, j, \ldots, k$, i.e. the degree of the polynomial, and the summation is taken from 1 to 3 for each index. The solution of the external Neumann problem for the sphere Σ under condition (2.6) will be

$$\varphi^{2k+1} = \frac{1}{a}\sum_{n=1}^{\infty}\frac{a^{n+2}}{(n+1)!\, r_1^{n+1}}Y_n = \sum_{n=1}^{\infty}\frac{a^{2n+1}}{(n-1)!\,(n+1)\, r_1^{n+1}}\sum_{(n)} u_{ij\ldots k}\nu_i\nu_j\ldots\nu_k =$$
$$= \sum_{n=1}^{\infty}\frac{a^{2n+1}}{(n-1)!\,(n+1)\, r_1^{2n+1}}\Big(\sum_{(n)} u_{ij\ldots k}\, y_i y_j \ldots y_k\Big) \tag{2.7}$$

Here $\mathbf{r}_1 = \mathbf{r} - \mathbf{r}_p$ is the radius-vector of the point P ($r_1 = a$ on Σ).

The potential φ° may be written in the form

$$\varphi^\circ = \mathbf{v}_0\mathbf{r} + \omega_1\Phi^1 + \omega_2\Phi^2 + \omega_3\Phi^3 \tag{2.8}$$

where ω_i is the projection of the vector $\boldsymbol{\omega}$ on the x_i (or y_i) axis, and Φ^i are functions harmonic in D (Zhukovskii potentials [5]) satisfying the following condition on S:

$$\partial\Phi^i / \partial n = (\mathbf{r} \times \mathbf{n})_i \qquad (i = 1, 2, 3)$$

The index i on the right-hand side denotes projection of the vector on the x_i-axis.

We estimate the order of the functions φ^k and their derivatives in terms of $\varepsilon = a/l$. Let the dimensions of the cavity be of the order of unit length and the distance l from P to S will be of the same order, so $l \sim 1$ and $a \sim \varepsilon$. Then φ° and all its derivatives are of the order $O(1)$. We find from (2.7) that far from the sphere Σ ($r_1 \sim l \sim 1$, in particular, on S) the function φ^1 and its derivatives have the order $O(\varepsilon^3)$. We find from the same formula that for $r_1 \sim a \sim \varepsilon$ (in particular, on Σ), $|\varphi^1| \sim \varepsilon$ and $|\nabla\varphi^1| \sim 1$. The function φ^2 is harmonic in D, the characteristic dimension of which is $O(1)$ and according to (2.3) has a normal derivative of the order of ε^3 on S. It follows from this that $|\varphi^2| \sim |\nabla\varphi^2| \sim \varepsilon^3$ everywhere in D. Analogously we get $|\varphi^{2k}| \sim |\nabla\varphi^{2k}| \sim \varepsilon^{3k}$ everywhere in D and $|\varphi^{2k+1}| \sim |\nabla\varphi^{2k+1}| \sim \varepsilon^{3k+3}$ far from Σ (for $r_1 \sim l$, in particular, on S), and near the sphere Σ (for $r_1 \sim a$, in particular, on Σ) we shall have

$$|\varphi^{2k+1}| \sim \varepsilon^{3k+1}, \qquad |\nabla\varphi^{2k+1}| \sim \varepsilon^{3k}$$

It follows from these estimates that $\varphi = \varphi^\circ + \varphi^1$ with an accuracy up to $O(\varepsilon^3)$ on the surface Σ. The function φ° is described on Σ by a segment of a Taylor series with center at the point P

$$\varphi^\circ = \varphi_p{}^\circ + \sum_i \varphi_i{}^\circ y_i + \tfrac{1}{2}\sum_{ij}\varphi_{ij}^\circ y_i y_j + O(\varepsilon^3) \tag{2.9}$$

Here $\varphi_p{}^\circ = \varphi^\circ(\mathbf{r}_p, t), \varphi_i{}^\circ, \varphi_{ij}{}^\circ$ are derivatives at P.

We find from (2.7) that

$$\varphi^1 = \tfrac{1}{2}\sum_i(\varphi_i{}^\circ - v_{pi})\, y_i + \tfrac{1}{3}\sum_{ij}\varphi_{ij}{}^\circ y_i y_j + O(\varepsilon^3) \tag{2.10}$$

on Σ with $k = 0$, taking account of the relation $u_i = \varphi_i{}^\circ - v_{pi}$, and where the summation is over all indices from 1 to 3.

By addition of (2.9) and (2.10) we get for φ on Σ

$$\varphi = \varphi_p{}^\circ + \tfrac{1}{2}\sum_i(3\varphi_i{}^\circ - v_{pi})\, y_i + \tfrac{5}{6}\sum_{ij}\varphi_{ij}{}^\circ y_i y_j + O(\varepsilon^3) \tag{2.11}$$

3. We transform the expression for kinetic energy of the fluid (1.10) by applying Green's theorem and considering that $\partial\varphi / \partial n = \partial\varphi^\circ / \partial n$ on S

$$T_2 = \frac{\rho}{2}\Big(\oint_S \varphi \frac{\partial\varphi^\circ}{\partial n}\, ds - \oint_\Sigma \varphi\frac{\partial\varphi}{\partial\nu}\, ds\Big) = \frac{\rho}{2}\Big[\oint_S \varphi\frac{\partial\varphi^\circ}{\partial n}\, ds - \oint_\Sigma \varphi\frac{\partial\varphi^\circ}{\partial\nu}\, ds + \oint_\Sigma \varphi\Big(\frac{\partial\varphi^\circ}{\partial\nu} - \frac{\partial\varphi}{\partial\nu}\Big)\, ds\Big] = \frac{\rho}{2}\Big\{\oint_S \varphi^\circ\frac{\partial\varphi^\circ}{\partial n}\, ds + \oint_\Sigma\Big[-\varphi^\circ\frac{\partial\varphi}{\partial\nu} + \varphi\Big(\frac{\partial\varphi^\circ}{\partial\nu} - \frac{\partial\varphi}{\partial\nu}\Big)\Big]\, ds\Big\} = T_2{}^\circ + T_2{}' \tag{3.1}$$

The first of the integrals in (3.1)

$$T_2{}^\circ = \frac{1}{2}\rho\oint_S \varphi^\circ\frac{\partial\varphi^\circ}{\partial n}\, ds \tag{3.2}$$

represents the kinetic energy of the fluid when it completely fills the cavity and is considered known. This integral is expressed in terms of the adjoined moments of inertia of the cavity [5], which may be calculated if Φ^1 is known.

We write the second integral of (3.1) in the form

$$T_2{}' = \frac{1}{2}\rho\oint_\Sigma\Big[-\varphi^\circ \mathbf{v}_p \boldsymbol{\nu} + \varphi\Big(\frac{\partial\varphi^\circ}{\partial\nu} - \mathbf{v}_p\boldsymbol{\nu}\Big)\Big]\, ds \tag{3.3}$$

taking into account (1.2).

We substitute here the expression for φ° from (2.9) and for φ from (2.11), and for $\partial\varphi^\circ/\partial\nu$ we put the Taylor series segment analogous to (2.9), with accuracy up to $O(\varepsilon^3)$, taking into account the relation $a\nu_i = y_i$ on Σ. Then the expression under the integral sign in (3.3) will reduce, to an accuracy of $\sim\varepsilon^3$, to a simple polynomial in y_i and the integral is readily calculated, giving

$$T_2' = \frac{3}{4}\rho\Omega\,(\nabla\varphi_p{}^\circ - \mathbf{v}_p)^2 - \frac{1}{2}\rho\Omega v_p{}^2 + O\,(\varepsilon^5) \qquad (3.4)$$

where the surface area Σ is if tne order of ε^2 and where $\nabla\varphi_p{}^\circ$ is the value of $\nabla\varphi^\circ$ at the point P , equal to

$$\nabla\varphi_p{}^\circ = \mathbf{v}_0 + \sum_s \omega_s \nabla\Phi_p{}^s \qquad (3.5)$$

in accordance with (2.8).

We write (1.10) in the form

$$T = T^\circ + \frac{3}{4}\rho\Omega\,(\nabla\varphi_p{}^\circ - \mathbf{v}_p)^2 + \frac{1}{2}(m - \rho\Omega)\,v_p{}^2 + O\,(\varepsilon^5) \qquad (3.6)$$

taking into account (3.1) and (3.4), where $T^\circ = T_1 + T_2{}^\circ$ is the kinetic energy of the body B when the cavity D is completely filled with fluid; this term is considered known.

Upon expressing $\mathbf{v}_p$ and $\nabla\varphi_p{}^\circ$ in terms of $\boldsymbol{\omega}$ by Formulas (1.3) and (3.5), and by differentiating according to (1.4), we find that

$$\mathbf{K} = \mathbf{K}^\circ + \frac{3}{2}\rho\Omega\sum_s \mathbf{e}_s\,[(\nabla\varphi_p{}^\circ - \mathbf{v}_p)\cdot\nabla\Phi_p{}^s] + $$
$$+ \frac{1}{2}\rho\Omega\,(3\nabla\varphi_p{}^\circ - \mathbf{v}_p)\times\mathbf{r}_p + m\mathbf{r}_p\times\mathbf{v}_p + O\,(\varepsilon^5) \qquad (3.7)$$

where $\mathbf{e}_s$ is a unit vector parallel to the x_s (or y_s) axis.

We pass over to establishment of the equation of motion for the sphere. We represent the function U in the region E by a Taylor series, omitting terms which after integration with respect to E in (1.1) will be of the order of ε^5 . (The volume Ω in the region E is of the order of ε^3).

From (1.1) we obtain

$$\Pi = \rho\int_D U dv - \rho\int_E (U_p + \mathbf{r}_1\cdot\nabla U_p)\,dv + O\,(\varepsilon^5)$$

The integral over E of the second term is equal to zero by virtue of the odd function $\mathbf{r}_1\cdot\nabla U_p$, and the integral on D does not depend on $\mathbf{R}_p$. Then, from (1.7)

$$\mathbf{Q}_p = \mathbf{F}_p + \rho\Omega\nabla U_p + O\,(\varepsilon^5) \qquad (3.8)$$

in which the second term is an Archimedes force.

We substitute (3.6) and (3.8) into (1.6), considering that T° is independent of the coordinates and velocity of the point P

$$\left(m + \frac{1}{2}\rho\Omega\right)\frac{d\mathbf{v}_p}{dt} - \frac{3}{2}\rho\Omega\left\{\frac{d}{dt}(\nabla\varphi_p{}^\circ) + [(\nabla\varphi_p{}^\circ - \mathbf{v}_p)\,\nabla]\,\nabla\varphi_p{}^\circ\right\} =$$
$$= \mathbf{F}_p + \rho\Omega\nabla U_p + O\,(\varepsilon^5) \qquad (3.9)$$

Differentiation with respect to t traces the trajectory of the point P, i.e.

$$\frac{d}{dt}\nabla\varphi^\circ(\mathbf{R}_p, t) = \frac{\partial}{\partial t}[\nabla\varphi^\circ(\mathbf{R}_p, t)] + (\mathbf{v}_p\nabla)\,\nabla\varphi^\circ(\mathbf{R}_p, t)$$

By taking this equality into account and omitting terms of the order of ε^5, we rewrite (3.9) in the final form

$$\left(m + \frac{1}{2}\rho\Omega\right)\frac{d\mathbf{v}_p}{dt} = \mathbf{F}_p + \rho\Omega\nabla U_p + \frac{3}{2}\rho\Omega\mathbf{w}_p{}^\circ \quad (3.10)$$

Here $\mathbf{w}_p{}^\circ$ is the acceleration of a fluid particle which would be at P if the cavity were completely filled, or in other words, the derivative of the velocity with respect to t along the trajectory of the fluid particle in the absence of the sphere E

$$\mathbf{w}_p{}^\circ = \frac{\partial(\nabla\varphi_p{}^\circ)}{\partial t} + (\nabla\varphi_p{}^\circ\nabla)\nabla\varphi_p{}^\circ \quad (3.11)$$

This quantity is subject to the equation of fluid motion

$$\rho\mathbf{w}^\circ = -\nabla p^\circ - \rho\nabla U$$

where p° is the fluid pressure with the cavity completely filled.

We write an explicit expression for $\mathbf{w}_p{}^\circ$ by means of the Zhukovskii potentials, differentiating (3.5) along the trajectory of the fluid particle

$$\mathbf{w}_p{}^\circ = \frac{d\mathbf{v}_0}{dt} + \sum_s \frac{d\omega_s}{dt}\nabla\Phi_p{}^s + \boldsymbol{\omega}\times\sum_s \omega_s\nabla\Phi_p{}^s + \\ + \sum_{sij}\omega_s\mathbf{e}_i\Phi_{ij}{}^s\left[\sum_k \omega_k\Phi_j{}^k - (\boldsymbol{\omega}\times\mathbf{r}_p)_j\right] \quad (3.12)$$

In Formula (3.12) all derivatives of Φ^s with respect to coordinates x_i are calculated at the point P. From (3.10) one may find equivalent forces due to hydrodynamic pressure

$$\mathbf{N} = m\frac{d\mathbf{v}_p}{dt} - \mathbf{F}_p = \rho\Omega\nabla U_p + \frac{3}{2}\rho\Omega\mathbf{w}_p{}^\circ - \frac{1}{2}\rho\Omega\frac{d\mathbf{v}_p}{dt}$$

Thus, the values of T, $\mathbf{K}$ and $\mathbf{Q}$ are determined by Formulas (3.6), (3.7) and (1.8) to an accuracy of $\sim\epsilon^5$, which guarantees high precision. Consequently, Equation (1.5) and (3.10) may be established with accuracy up to ϵ^5 (the potential Φ° and values of T°, $\mathbf{Q}^\circ$ and $\mathbf{K}^\circ$ are considered known, as stated above). The equations of motion obtained are evidently equivalent to the equations of motion of a system comprising a solid body and a material point P acting on each other.

The difference $T - T^\circ$ has the order of $\epsilon^3(\Omega\sim a^3)$, i.e. the order of the ratio of the volume of the sphere E to the cavity D. This is just the order of perturbances in the equations of motion of the body with a fluid on account of the sphere E (if the masses of the body and fluid are of the same order). Hence, without loss of accuracy the equations of motion may be integrated as follows. First we solve the equations of motion of the body when the cavity is completely filled with fluid; i.e. we determine the undisturbed motion. The we integrate Equation (3.10) for motion of the sphere E, assuming that the body motion is undisturbed. Afterwards, we substitute the coordinates and velocities of the point P in Equation (1.5) and determine the disturbed motion of the body with fluid.

4. We consider certain examples. Let the fluid in the cavity be in translational motion ($\nabla\varphi^{\circ} \equiv \mathbf{v}_0$), in absence of the sphere E, this may occur either with translational motion of the body ($\boldsymbol{\omega} = 0$), or in the case of a spherical cavity (all $\Phi^i = 0$). In this case the fluid, which completely fills the cavity, is equivalent to a material mass point μ at the center of inertia of the fluid. Assuming that $\mathbf{F}_p = 0$ and $U = 0$, we find from (3.10) and (3.12)

$$\frac{d\mathbf{v}_p}{dt} = \frac{3\rho\Omega}{\rho\Omega + 2m}\frac{d\mathbf{v}_0}{dt}, \qquad \mathbf{v}_p = \frac{3\rho\Omega}{\rho\Omega + 2m}\mathbf{v}_0 + \mathbf{c} \tag{4.1}$$

where $\mathbf{c}$ is an arbitrary constant vector.

This result may also be easily obtained from the theory of added mass.

In particular, for a bubble ($m = 0$)

$$d\mathbf{v}_p/dt = 3d\mathbf{v}_0 / dt$$

i.e. the absolute acceleration is three times the absolute cavity acceleration [6].

By substitution of (4.1) in (3.6) we find, with accuracy to an inessential constant,

$$T = T^{\circ} + \frac{3\,(m - \rho\Omega)\,\rho\Omega}{2\,(\rho\Omega + 2m)}\,v_0^2 \tag{4.2}$$

Thus, the presence of the sphere E in the fluid (for translational motion) is equivalent to a change in the fluid mass by a constant value. If the sphere E is a bubble, then the equivalent mass of the cavity with the fluid and bubble is equal to $\mu - 3\rho\Omega$.

We study still another example: the motion of the sphere E in an ellipsoidal cavity

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

Let $\mathbf{v}_0 = 0$, $\mathbf{F}_p = 0$ and $U = 0$, and the motion of the cavity be given as a uniform rotation about an immovable axis passing through a point O – the center of symmetry of the ellipsoid. On these assumptions Equation (3.10) together with (3.12) take the form

$$\mathbf{r}'' + 2\boldsymbol{\omega} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) =$$

$$= \alpha \left\{\boldsymbol{\omega} \times \sum_s \omega_s \nabla \Phi^s + \sum_{sij} \omega_s \mathbf{e}_i \Phi_{ij}^s \left[\sum_k \omega_k \Phi_j^k - (\boldsymbol{\omega} \times \mathbf{r})_j\right]\right\} \tag{4.3}$$

$$\alpha = 3\rho\Omega / (\rho\Omega + 2m) \quad (0 \leqslant \alpha \leqslant 3) \tag{4.4}$$

Here the absolute acceleration of the point P is expressed by a transfer, a relative and a Coriolis acceleration, and the index p is omitted everywhere. The Zhukovskii potentials for an ellipsoidal cavity are known to be [5]

$$\Phi^3 = k_3 x_1 x_2, \qquad k_3 = (a_1^2 - a_2^2) / (a_1^2 + a_2^2) \qquad (|k_3| \leqslant 1) \tag{4.5}$$

The remaining Φ^i are obtained from (4.5) by transposing indices. Upon substitution of (4.5) into (4.3) we obtain a linear system with constant coefficients describing the motion of the point P.

For simplicity we assume the cavity to rotate about its axis of symmetry ($\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$). Then, after substitution of (4.5) into (4.3) and taking projection on the x_1-axes we obtain

$$\ddot{x}_1 - 2\omega\dot{x}_2 - \omega^2 x_1 = \alpha\omega^2 k(k - 2)\,x_1 \qquad (k = k_3) \tag{4.6}$$

$$\ddot{x}_2 + 2\omega\dot{x}_1 - \omega^2 x_2 = \alpha\omega^2 k(k + 2) x_2, \qquad \ddot{x}_3 = 0$$

It is evident from (4.6) that motion of the point P along the axis of rotation x_3 proceeds by inertia. Therefore, as it was to be expected, the positions of equilibrium of the sphere on the axis of rotation ($x_1 = x_2 = 0$,

$x_3 = x_3^\circ$) are unstable, and there are no other equilibrium positions of the system (4.6).

We consider motion of the point P in the plane rotation $x_1 x_2$. An elementary investigation of the characteristic equation for the first pair of (4.6) permits one to obtain the stability conditions for the equilibrium position $x_1 = x_2 = 0$ in the $x_1 x_2$ plane. These conditions (necessary and sufficient) have the form

$$(2|k| - k^2)^{-1} > \alpha > 1, \qquad 0 < |k| < 1 \tag{4.7}$$

For fulfillment of condition (4.7) the characteristic equation has no multiple pure imaginary roots (no dissipation in the system) and the sphere will not depart from the axis of rotation. By transformation of (4.7) and substitution of α and k from (4.4) and (4.5) and assuming that $a_1 \geqslant a_2$, we get

$$1 > \frac{m}{\rho\Omega} > 1 - \frac{6a_2^4}{(a_1^2 + a_2^2)^2}, \qquad a_1 > a_2 \tag{4.8}$$

For a bubble ($m = 0$) we obtain from (4.8) the stability condition for equilibrium in the plane of rotation in the form

$$1 < a_1 / a_2 < (\sqrt{6} - 1)^{1/2} \approx 1.20$$

5. We consider the reaction of the solid body with a cavity containing fluid and an air bubble to the action of shock. At a certain instant of time let there be applied to the body impulse (shock) forces and moments exciting instantaneous changes in the values of $\mathbf{v}_0$, $\boldsymbol{\omega}$ and $\mathbf{v}_p$, as $\delta\mathbf{v}_0$, $\delta\boldsymbol{\omega}$ and $\delta\mathbf{v}_p$. At the same time, T, $\mathbf{K}$ and $\mathbf{Q}$ also receive finite increments δT, $\delta\mathbf{K}$ and $\delta\mathbf{Q}$. Here the bubble can not be considered as an undeformable sphere. The effect of impulse forces on the surface tension may be neglected, as well as the effects on other forces limited in value.

The hydrodynamic potential is obtained at the moment of shock as a finite increment $\delta\varphi(\mathbf{r})$. The function $\delta\varphi$ is harmonic in D_1 and satisfies the boundary conditions (C is an arbitrary constant)

$$\partial\delta\varphi / \partial n = (\delta\mathbf{v}_0 + \delta\boldsymbol{\omega} \times \mathbf{r})\,\mathbf{n} \quad \text{on } S, \qquad \delta\varphi = C \quad \text{on } \Sigma \tag{5.1}$$

The second condition of (5.1) expresses the absence of impulse action on the surface Σ.

The boundary problem for $\delta\varphi$ may be solved by the Schwarz method, on the assumption, analogous to (2.1)

$$\delta\varphi = \delta\varphi^\circ + \delta\varphi^1 + \delta\varphi^2 + \ldots \tag{5.2}$$

The functions $\delta\varphi^{2k}$ are harmonic in D and $\delta\varphi^{2k+1}$ outside of E, while $\delta\varphi^\circ$ satisfies the first condition of (5.1) and the remaining functions are subjected to the conditions

$$\frac{\partial\delta\varphi^{2k}}{\partial n} = -\frac{\partial\delta\varphi^{2k-1}}{\partial n} \text{ on } S, \qquad \delta\varphi^{2k+1} = -\delta\varphi^{2k} + \delta\varphi_p^{\ 2k} \text{ on } \Sigma \tag{5.3}$$

for $\delta\varphi^{2k}$ and for $\delta\varphi^{2k+1}$, respectively. The index p refers to the value at the point P and the constant terms $\delta\varphi_p^{\ 2k}$ are chosen for convenience. The functions $\delta\varphi^i$ have the same order with respect to ϵ as φ^i in Section 2.

The potential $\delta\varphi^\circ$ is determined by an equation analogous to (2.8)

$$\delta\varphi^\circ = \delta\mathbf{v}_0\mathbf{r} + \delta\omega_1\Phi^1 + \delta\omega_2\Phi^2 + \delta\omega_3\Phi^3$$

We write $\delta\varphi^1$, first expanding condition (5.3) for $k = 1$ in a Taylor series centered at P

$$\delta\varphi^1 = -\sum_i \delta\varphi_i^\circ y_i - \frac{1}{2!}\sum_{ij}\delta\varphi_{ij}^\circ y_i y_j - \ldots \tag{5.4}$$

Here $\delta\varphi_i^\circ$, $\delta\varphi_{ij}^\circ$, etc., denote corresponding derivatives at P. The solution to the external Dirichlet problem for the sphere E with boundary condition expressed by (5.4) is given by a formula analogous to (2.7)

$$\delta\varphi^1 = -\sum_{n=1}^{\infty} \frac{a^{2n+1}}{n!\,r_1^{2n+1}} \Big(\sum_{(n)} \delta\mathring{\varphi}_{ij\ldots k} y_i y_j \ldots y_k \Big) \tag{5.5}$$

We calculate the increment in kinetic energy of the fluid from Formula

$$\delta T_2 = \frac{1}{2}\rho \int_{D_1} [(\nabla\varphi + \nabla\delta\varphi)^2 - (\nabla\varphi)^2]\, dv = \frac{1}{2}\rho \int_{D_1} (2\nabla\varphi + \nabla\delta\varphi)\nabla\delta\varphi\, dv$$

By transforming this expression analogously to (3.1), and by use of Green's theorem and the equalities

$$\frac{\partial\delta\varphi}{\partial n} = \frac{\partial\delta\varphi^\circ}{\partial n} \quad \text{on } S, \qquad \delta\varphi = C \quad \text{on } \Sigma$$

we arrive at Formula

$$\delta T_2 = \delta T_2^\circ - \frac{1}{2}\rho \oint_\Sigma (2\varphi^\circ + \delta\varphi^\circ)(\mathbf{v}\nabla\delta\varphi)\, ds \tag{5.6}$$

where δT_2° is the increment in T_2° from (3.2).

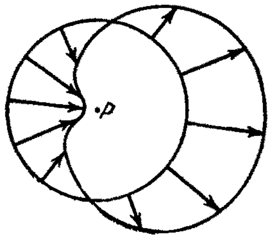

Fig. 2

We find the function

$$\nabla\delta\varphi = \nabla\delta\varphi^\circ + \nabla\delta\varphi^1 + O(\varepsilon^3)$$

which enters into the integral of (5.6), by expansion $\nabla\delta\varphi^\circ$ in a Taylor series analogous to (2.9) and calculation $\nabla\delta\varphi^1$ by differentiation of (5.5).

After such calculations we obtain on the surface Σ

$$\nabla\delta\varphi = \frac{\mathbf{r}_1}{a^2}\Big[3\sum_i \delta\varphi_i^\circ y_i + \frac{5}{2}\sum_{ij} \delta\varphi_{ij}^\circ y_i y_j + \frac{7}{6}\sum_{ijk} \delta\mathring{\varphi}_{ijk} y_i y_j y_k\Big] + O(\varepsilon^3) \tag{5.7}$$

The nature of the deformation of a spherical bubble under shock is seen. Points on the surface Σ acquire an additional velocity from the shock directed along the radius of the sphere and equal as a first approximation to $3u \cdot \cos\theta$ ($\mathbf{u}$ is the additional velocity which would be obtained for the same shock on the fluid particle at P inside a completely filled cavity, θ is the angle measured from the direction of $\mathbf{u}$). Fig.2 shows a diagram of the velocities acquired on the bubble surface during shock. Such a distribution agrees with the results of exprrimental studies of the motion of a bubble in a fluid (see, e.g. [7]).

By use of (5.7), and by representing φ° by Formula (2.9) and $\delta\varphi^\circ$ by an analogous formula, it is not difficult to calculate the integral in (5.6), after which the increment in kinetic energy of the whole system may be wtitten in the form

$$\delta T = \delta T^\circ - {}^3/_2\rho\Omega\,(2\nabla\varphi_p^\circ + \nabla\delta\varphi_p^\circ)\nabla\delta\varphi_p^\circ + O(\varepsilon^5) \tag{5.8}$$

We now consider the effect of a shock in the case where the bubble remains a rigid undeformable sphere with mass $m = 0$. For this one may use Formulas of Section 3. Substitute for the derivatives of the velocities with time in Equations (3.10) and (3.12) the finite increments at the moment of shock, and neglect the impulses of the forces $\mathbf{F}_p$, $\rho\Omega\nabla U$ at that interval of time. Then we obtain

$$\delta\mathbf{v}_p = 3\nabla\delta\varphi_p^\circ = 3\,(\delta\mathbf{v}_0 + \sum \delta\omega_s \nabla\Phi_p^s) \tag{5.9}$$

It is not difficult now, by using (5.9) and (3.6), to calculate the increment δT at the moment of shock. After a simple calculation we obtain an expression of an accuracy comparable with (5.8). Consequently, with accuracy up to terms in ε^5, the behavior of the system in question during shock does not depend on whether the initially spherical bubble is considered to be deformable (with a free surface) or rigid; then the increments $\delta\mathbf{Q}$, $\delta\mathbf{K}$ may be calculated with an accuracy up to ε^5 by Formulas (1.8) and (3.7), taking into account (5.9) and setting $m = 0$. The calculation gives

$$\delta \mathbf{Q} = \delta \mathbf{Q}^\circ - 3\rho\Omega\nabla\delta\varphi_p^\circ, \qquad \delta \mathbf{K} = \delta \mathbf{K}^\circ - 3\rho\Omega \sum_s \mathbf{e}_s (\nabla\Phi_p^s \cdot \nabla\delta\varphi_p^\circ)$$

These formulas allow the behavior of the system during shock to be calculated.

6. We note that Equation (3.10) holds for an accuracy of $\sim \varepsilon^2 \sim a^2 l^{-2}$ also in the case where the sphere E moves in any potential flow with a characteristic dimension l , if $l \gg a$ (not necessarily inside certain cavities). By making use of the equation of fluid motion we write Equation (3.10) in the form

$$(m + {}^1/_2\, \rho\Omega)\, d\mathbf{v}_p / dt = \mathbf{F}_p - {}^1/_2\, \rho\Omega\nabla U - {}^3/_2\, \Omega\nabla p^\circ \tag{6.1}$$

where p° as before is the pressure at P in the undisturbed flow, i.e. in the absence of the sphere. Through Equation (6.1) one may study the motion of a small rigid sphere in the arbitrary potential flow of an ideal incompressible fluid.

Let the undisturbed motion of the fluid be established, the force $\mathbf{F}_p$ be a potential force ($\mathbf{F}_p = -\nabla W$), and let W and U be independent of the time. Then the equation of motion for the sphere E has a first integral

$${}^1/_2\,(m + {}^1/_2\,\rho\Omega)\, v_p^2 + \Phi = C, \qquad \Phi = W + {}^1/_2\,\rho\Omega\, U + {}^3/_2\, \Omega p^\circ$$

where Φ plays the role of potential energy and where p° may be expressed by the velocity $\mathbf{v}^\circ$ of the undisturbed flow by means of the Bernoulli integral.

The equilibrium positions of the sphere in the flow correspond to stationary points of the function Φ (where $\nabla\Phi = 0$), and their stability is determined by the character of the stationary points. Let the potential of the external forces be harmonic functions ($\Delta W = \Delta U = 0$) such as gravity force, for example . It is shown on p.62 of [6] that in this case $\Delta p^\circ \leqslant 0$, everywhere and consequently $\Delta\Phi \leqslant 0$, i.e. Φ is a superharmonic function. It is known that the minimum of such functions is achieved on the boundary of the region [3] and that internal stationary points are not strict minima. Therefore, for the assumptions made, one must expect in the majority of cases to have an instability in the equilibrium position of the sphere inside the steady potential flow of an ideal incompressible fluid.

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